

On the α -Amenability of Hypergroups

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Abstract

Let $UC(K)$ denote the Banach space of all bounded uniformly continuous functions on a hypergroup K . The main results of this article concern on the α -amenability of $UC(K)$ and quotients and products of hypergroups. It is also shown that a Sturm-Liouville hypergroup with a positive index is α -amenable if and only if $\alpha = 1$.

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1 Introduction

In [20] M. Skantharajah systematically studied the amenability of hypergroups. Among other things, he obtained various equivalent statements on the amenability of hypergroups. Let K be a locally compact hypergroup. Let $L^1(K)$ and $UC(K)$ denote the hypergroup algebra and the Banach space of all bounded uniformly continuous functions on K , respectively. He showed that K is amenable if and only if there exists an invariant mean on $UC(K)$. Observe that, contrary to the group case, $UC(K)$ may fail to be an algebra in general. Commutative or compact hypergroups are amenable, and the amenability of $L^1(K)$ implies the amenability of K ; however, the converse is not valid any longer even if K is commutative (see also [2, 7]).

Recently the notion of α -amenable hypergroups was introduced and studied in [7]. Suppose that K is commutative, let $\alpha \in \widehat{K}$, and denote by $I(\alpha)$ the maximal ideal in $L^1(K)$ generated by α . As shown in [7], K is α -amenable if and only if either $I(\alpha)$ has a bounded approximate identity or K satisfies the modified Reiter condition of P_1 -type in α . The latter condition together with the recursion formulas for orthogonal polynomials yields a sufficient condition for the α -amenability of a polynomial hypergroup. However, this condition is not available for well known hypergroups on the non-negative real axis.

The purpose of this article is to generalize the notion of α -amenability for K to the Banach space $UC(K)$. It then turns out that the α -amenability of K is equivalent to the α -amenability of $UC(K)$,

and a α -mean on $UC(K)$ is unique if and only if α belongs to $L^1(K) \cap L^2(K)$. Furthermore, some results are obtained on the α -amenability of quotients and products of hypergroups. Given a Sturm-Liouville hypergroup K with a positive index, it is also shown that there exist non-zero point derivations on $L^1(K)$. Therefore, $L^1(K)$ is not weakly amenable, $\{\alpha\}$ ($\alpha \neq 1$) is not a spectral set, and K is not α -amenable if $\alpha \neq 1$. However, an example (consisting of a certain Bessel-Kingman hypergroup) shows that in general K is not necessarily α -amenable if $\{\alpha\}$ is a spectral set.

This article is organized as follows: Section 2 collects pertinent concepts concerning on hypergroups. Section 3 considers the α -amenability of $UC(K)$. Section 4 contains the α -amenability of quotients and products of hypergroups, and Section 5 is considered on the question of α -amenability of Sturm-Liouville hypergroups.

2 Preliminaries

Let (K, ω, \sim) be a locally compact hypergroup, where $\omega : K \times K \rightarrow M^1(K)$ defined by $(x, y) \mapsto \omega(x, y)$, and $\sim : K \rightarrow K$ defined by $x \mapsto \tilde{x}$, denote the convolution and involution on K , where $M^1(K)$ stands for all probability measures on K . K is called commutative if $\omega(x, y) = \omega(y, x)$, for every $x, y \in K$.

Throughout the article K is a commutative hypergroup. Let $C_c(K)$, $C_0(K)$, and $C^b(K)$ be the spaces of all continuous functions, those which have compact support, vanishing at infinity, and bounded on K respectively; both $C^b(K)$ and $C_0(K)$ will be topologized by the uniform norm $\|\cdot\|_\infty$. The space of complex regular Radon measures on K will be denoted by $M(K)$, which coincides with the dual space of $C_0(K)$ [11, Riesz's Theorem (20.45)]. The translation of $f \in C_c(K)$ at the point $x \in K$, $T_x f$, is defined by $T_x f(y) = \int_K f(t) d\omega(x, y)(t)$, for every $y \in K$. Being K commutative ensures the existence of a Haar measure on K which is unique up to a multiplicative constant [21]. Thus, according to the translation T , let m be the Haar measure on K , and let $(L^1(K), \|\cdot\|_1)$ denote the usual Banach space of all integrable functions on K [12, 6.2]. For $f, g \in L^1(K)$ the convolution and involution are defined by $f * g(x) := \int_K f(y) T_{\tilde{y}} g(x) dm(y)$ (m -a.e. on K) and $f^*(x) = \overline{f(\tilde{x})}$, respectively, that $(L^1(K), \|\cdot\|_1)$ becomes a commutative Banach $*$ -algebra. If K is discrete, then $L^1(K)$ has an identity; otherwise $L^1(K)$ has a b.a.i. (bounded approximate identity), i.e. there exists a net $\{e_i\}_i$ of functions in $L^1(K)$ with $\|e_i\|_1 \leq M$, for some $M > 0$, such that $\|f * e_i - f\|_1 \rightarrow 0$ as $i \rightarrow \infty$ [3]. The dual space $L^1(K)^*$ can be identified with the space $L^\infty(K)$ of essentially bounded Borel measurable complex-valued functions on K . The bounded multiplicative linear functionals on $L^1(K)$ can be identified with

$$\mathfrak{X}^b(K) := \left\{ \alpha \in C^b(K) : \alpha \neq 0, \omega(x, y)(\alpha) = \alpha(x)\alpha(y), \forall x, y \in K \right\},$$

where $\mathfrak{X}^b(K)$ is a locally compact Hausdorff space with the compact-open topology. $\mathfrak{X}^b(K)$ with its subset

$$\widehat{K} := \left\{ \alpha \in \mathfrak{X}^b(K) : \alpha(\tilde{x}) = \overline{\alpha(x)}, \forall x \in K \right\}$$

are considered as the character spaces of K .

The Fourier-Stieltjes transform of $\mu \in M(K)$, $\widehat{\mu} \in C^b(\widehat{K})$, is $\widehat{\mu}(\alpha) := \int_K \overline{\alpha(x)} d\mu(x)$, which by restriction on $L^1(K)$ it is called Fourier transform and $\widehat{f} \in C_0(\widehat{K})$, for every $f \in L^1(K)$.

There exists a unique regular positive Borel measure π on \widehat{K} with the support \mathcal{S} such that

$$\int_K |f(x)|^2 dm(x) = \int_{\mathcal{S}} |\widehat{f}(\alpha)|^2 d\pi(\alpha),$$

for all $f \in L^1(K) \cap L^2(K)$. π is called Plancherel measure and its support, \mathcal{S} , is a nonvoid closed subset of \widehat{K} . Observe that the constant function 1 is in general not contained in \mathcal{S} . We have $\mathcal{S} \subseteq \widehat{K} \subseteq \mathfrak{X}^b(K)$, where proper inclusions are possible; see [12, 9.5].

3 α -Amenability of $UC(K)$

Definition 3.1. Let K be a commutative hypergroup and $\alpha \in \widehat{K}$. Let X be a subspace of $L^\infty(K)$ with $\alpha \in X$ which is closed under complex conjugation and is translation invariant. Then X is called α -amenable if there exists a $m_\alpha \in X^*$ with the following properties:

- (i) $m_\alpha(\alpha) = 1$,
- (ii) $m_\alpha(T_x f) = \alpha(x)m_\alpha(f)$, for every $f \in X$ and $x \in K$.

The hypergroup K is called α -amenable if $X = L^\infty(K)$ is α -amenable; in the case $\alpha = 1$, K respectively $L^\infty(K)$ is called amenable. As shown in [7], K is α -amenable if and only if either $I(\alpha)$ has a b.a.i. or K has the modified Reiter's condition of P_1 type in the character α . The latter is also equivalent to the α -left amenability of $L^1(K)$, if α is real-valued [2]. For instance, commutative hypergroups are amenable, and compact hypergroups are α -amenable for every character α .

If K is a locally compact group, then the amenability of K is equivalent to the amenability of diverse subalgebras of $L^\infty(K)$, e.g. $UC(K)$ the algebra of bounded uniformly continuous functions on K [17]. The same is true for hypergroups although $UC(K)$ fails to be an algebra in general [20]. We now prove this fact in terms of α -amenability.

Let $UC(K) := \{f \in C^b(K) : x \mapsto T_x f \text{ is continuous from } K \text{ to } (C^b(K), \|\cdot\|_\infty)\}$. The function space $UC(K)$ is a norm closed, conjugate closed, translation invariant subspace of $C^b(K)$ containing the constants and the continuous functions vanishing at infinity [20, Lemma 2.2]. Moreover, $\mathfrak{X}^b(K) \subset UC(K)$ and $UC(K) = L^1(K) * L^\infty(K)$. Let B be a subspace of $L^\infty(K)$ such that $UC(K) \subseteq B$. K is amenable if and only if B is amenable [20, Theorem 3.2].

The following theorem provides a further equivalent statement to the α -amenability of K .

Theorem 3.2. Let $\alpha \in \widehat{K}$. Then $UC(K)$ is α -amenable if and only if K is α -amenable.

Proof: Let $UC(K)$ be α -amenable. There exists a $m_\alpha^{uc} \in UC(K)^*$ such that $m_\alpha^{uc}(\alpha) = 1$ and $m_\alpha^{uc}(T_x f) = \alpha(x)m_\alpha^{uc}(f)$ for all $f \in UC(K)$ and $x \in K$. Let $g \in L^1(K)$ such that $\widehat{g}(\alpha) = 1$. Define

$m_\alpha : L^\infty(K) \longrightarrow \mathbb{C}$ by

$$m_\alpha(\varphi) = m_\alpha^{uc}(\varphi * g) \quad (\varphi \in L^\infty(K))$$

that $m_\alpha|_{UC(K)} = m_\alpha^{uc}$. Since $\varphi * g \in UC(K)$, m_α is a well-defined bounded linear functional on $L^\infty(K)$, $m_\alpha(\alpha) = 1$, and for all $x \in K$ we have

$$\begin{aligned} m_\alpha(T_x \varphi) &= m_\alpha^{uc}((T_x \varphi) * g) \\ &= m_\alpha^{uc}((\delta_{\tilde{x}} * \varphi) * g)) \\ &= m_\alpha^{uc}(\delta_{\tilde{x}} * (\varphi * g)) \\ &= m_\alpha^{uc}(T_x(\varphi * g)) \\ &= \alpha(x)m_\alpha^{uc}(\varphi * g) \\ &= \alpha(x)m_\alpha(\varphi). \end{aligned} \tag{1}$$

The latter shows that every α -mean on $UC(K)$ extends on $L^\infty(K)$. Plainly the restriction of any α -mean of K on $UC(K)$ is a α -mean on $UC(K)$. Therefore, the statement is valid. \square

Corollary 3.3. Let $\alpha \in \widehat{K}$ and $UC(K) \subseteq B \subseteq L^\infty(K)$. Then K is α -amenable if and only if B is α -amenable.

The Banach space $L^\infty(K)^*$ with the Arens product defined as follows is a Banach algebra:

$$\langle m \cdot m', f \rangle = \langle m, m' \cdot f \rangle, \text{ in which } \langle m' \cdot f, g \rangle = \langle m', f \cdot g \rangle,$$

and $\langle f \cdot g, h \rangle = \langle f, g * h \rangle$ for all $m, m' \in L^\infty(K)^*$, $f \in L^\infty(K)$ and $g, h \in L^1(K)$ where $\langle f, g \rangle := f(g)$.

The Banach space $UC(K)^*$ with the restriction of the Arens product is a Banach algebra, and it can be identified with a closed right ideal of the Banach algebra $L^1(K)^{**}$ [14].

If $m, m' \in UC(K)^*$, $f \in UC(K)$, and $x \in K$, then $m' \cdot f \in UC(K)$ and we may have

$$\langle m \cdot m', f \rangle = \langle m, m' \cdot f \rangle, \langle m' \cdot f, x \rangle = \langle m', T_x f \rangle.$$

If $y \in K$, then $\int_K T_t f d\omega(x, y)(t) = T_y(T_x f)$ which implies that

$$T_x(m \cdot f) = m \cdot T_x f,$$

as

$$\begin{aligned} T_x(m \cdot f)(y) &= \int_K m \cdot f(t) d\omega(x, y)(t) \\ &= \int_K \langle m, T_t f \rangle d\omega(x, y)(t) \\ &= \langle m, \int_K T_t f d\omega(x, y)(t) \rangle. \end{aligned}$$

Let $X = UC(K)$, $f \in X$ and $g \in C_c(K)$ ($g \geq 0$) with $\|g\|_1 = 1$. Since the mapping $x \rightarrow T_x f$ is continuous from K to $(C^b(K), \|\cdot\|_\infty)$ and the point evaluation functionals in X^* separate points of X , we have

$$g * f = \int_K g(x) T_{\bar{x}} f dm(x).$$

Therefore, for every 1-mean m on X we have $m(f) = m(g * f)$. Hence, two 1-means m and m' on $L^\infty(K)$ are equal if they are equal on $UC(K)$, as

$$m(f) = m(g * f) = m'(g * f) = m'(f) \quad (f \in L^\infty(K))$$

and g is assumed as above. The latter together with [20, Theorem 3.2] show a bijection between means on $UC(K)$ and $L^\infty(K)$. So, if $UC(K)$ is amenable with a unique mean, then its extension on $L^\infty(K)$ is also unique which implies that K is compact [13], therefore the identity character is isolated in \mathcal{S} , the support of the Plancherel measure. We have the following theorem in general:

Theorem 3.4. If $UC(K)$ is α -amenable with the unique α -mean m_α^{uc} , then $m_\alpha^{uc} \in L^1(K) \cap L^2(K)$, $\{\alpha\}$ is isolated in \mathcal{S} and $m_\alpha^{uc} = \frac{\pi(\alpha)}{\|\alpha\|_2^2}$, where $\pi : L^1(K) \rightarrow L^1(K)^{**}$ is the canonical embedding. If α is positive, then K is compact.

Proof: Let m_α^{uc} be the unique α -mean on $UC(K)$, $n \in UC(K)^*$ and $\{n_i\}_i$ be a net converging to n in the w^* -topology. Let $x \in K$ and $f \in UC(K)$. Then

$$\begin{aligned} \langle m_\alpha^{uc} \cdot n_i, T_x f \rangle &= \langle m_\alpha^{uc}, n_i \cdot (T_x f) \rangle \\ &= \langle m_\alpha^{uc}, T_x(n_i \cdot f) \rangle \\ &= \alpha(x) \langle m_\alpha^{uc}, n_i \cdot f \rangle \\ &= \alpha(x) \langle m_\alpha^{uc} \cdot n_i, f \rangle. \end{aligned} \tag{2}$$

For $\lambda_i = \langle n_i, \alpha \rangle \neq 0$, since the associated functional to the character α on $UC(K)^*$ is multiplicative [27], $m_\alpha^{uc} \cdot n_i / \lambda_i$ is a α -mean on $UC(K)$ which is equal to m_α^{uc} . Then the mapping $n \rightarrow m_\alpha^{uc} \cdot n$ defined on $UC(K)^*$ is $w^* \text{-} w^*$ continuous, hence m_α^{uc} is in the topological centre of $UC(K)^*$, i.e. $M(K)$; see [13, Theorem 3.4.3]. Since $\widehat{m_\alpha^{uc}}(\beta) = \delta_\alpha(\beta)$ and $\widehat{m_\alpha^{uc}} \in C^b(\widehat{K})$, $\{\alpha\}$ is an open-closed subset of \widehat{K} . The algebra $L^1(K)$ is a two-sided closed ideal in $M(K)$ and the Fourier transform is injective, so m_α^{uc} and α belong to $L^1(K) \cap L^2(K)$. The inversion theorem, [3, Theorem 2.2.36], indicates $\alpha = \widehat{\alpha}$, accordingly $\alpha \in \mathcal{S}$. Let $m_\alpha := \pi(\alpha) / \|\alpha\|_2^2$. Obviously m_α is a α -mean on $L^\infty(K)$, and the restriction of m_α on $UC(K)$ yields the desired unique α -mean.

If α is positive, then

$$\alpha(x) \int_K \alpha(t) dm(t) = \int_K T_x \alpha(t) dm(t) = \int_K \alpha(t) dm(t)$$

which implies that $\alpha = 1$, hence K is compact. \square

Observe that in contrast to the case of locally compact groups, there exist noncompact hypergroups with unique α -means. For example, for little q-Legendre polynomial hypergroups, we have $\widehat{K} \setminus \{1\} \subset L^1(K) \cap L^2(K)$; see [8]. Therefore, by Theorem 3.4, $UC(K)$ and K are α -amenable with the unique α -mean m_α , whereas K has infinitely many 1-means [20].

Let $\Sigma_\alpha(X)$ be the set of all α -means on $X = L^\infty(K)$ or $UC(K)$. If $\alpha = 1$, then $\Sigma_1(X)$ is nonempty (as K is commutative) weak*-compact convex set in X^* [20]. If $\alpha \neq 1$ and X is α -amenable, then the same is true for $\Sigma_\alpha(X)$.

Theorem 3.5. Let X be α -amenable ($\alpha \neq 1$). Then $\Sigma_\alpha(X)$ is a nonempty weak*-compact convex subset of X^* . Moreover, $\Sigma_\alpha(X) \cdot M/\langle M, \alpha \rangle \subseteq \Sigma_\alpha(X)$, for all $M \in X^*$ with $\langle M, \alpha \rangle \neq 0$. Furthermore, if $m_\alpha \in \Sigma_\alpha(X)$, then $m_\alpha^n = m_\alpha$ for all $n \in \mathbb{N}$.

Proof: Let $0 \leq \lambda \leq 1$ and $m_\alpha, m'_\alpha \in \Sigma_\alpha(X)$. If $m''_\alpha := \lambda m_\alpha + (1 - \lambda)m'_\alpha$, then $m''_\alpha(\alpha) = 1$ and $m''_\alpha(T_x f) = \alpha(x)m''_\alpha(f)$, for every $f \in X$ and $x \in K$; hence $m''_\alpha \in \Sigma_\alpha(X)$.

If $\{m_i\} \subset \Sigma_\alpha(X)$ such that $m_i \xrightarrow{w^*} m$, then $m \in \Sigma_\alpha(X)$. We have $m(\alpha) = 1$ and

$$m(T_x f) = \lim_{i \rightarrow \infty} m_i(T_x f) = \alpha(x) \lim_{i \rightarrow \infty} m_i(f) = \alpha(x)m(f),$$

for all $f \in X$ and $x \in K$. Moreover, $\Sigma_\alpha(X)$ is w^* -compact by Alaoglu's theorem [6, p.424]. Let $M \in X^*$ with $\lambda = \langle M, \alpha \rangle \neq 0$. Then $M' := m_\alpha \cdot M/\lambda$ is a α -mean on X , as

$$\langle m_\alpha \cdot M, T_x f \rangle = \langle m_\alpha, M \cdot T_x f \rangle = \langle m_\alpha, T_x(M \cdot f) \rangle = \alpha(x) \langle m_\alpha, M \cdot f \rangle = \alpha(x) \langle m_\alpha \cdot M, f \rangle.$$

Since $g \cdot m_\alpha = m_\alpha \cdot g = \widehat{g}^*(\alpha)m_\alpha$ for all $g \in L^1(K)$, the continuity of the Arens product in the first variable on X together with Goldstein's theorem yield $m_\alpha^2 = m_\alpha$; hence, $m_\alpha^n = m_\alpha$ for all $n \in \mathbb{N}$. \square

Remark 3.6. Observe that if K is α and β -amenable, then $m_\alpha \cdot m_\beta = m_\beta \cdot m_\alpha$ if and only if $m_\alpha \cdot m_\beta = \delta_\alpha(\beta)m_\alpha$, as

$$\begin{aligned} \alpha(x) \langle m_\alpha \cdot m_\beta, f \rangle &= \alpha(x) \langle m_\alpha, m_\beta \cdot f \rangle \\ &= \langle m_\alpha, T_x(m_\beta \cdot f) \rangle \\ &= \langle m_\alpha, m_\beta \cdot T_x f \rangle \\ &= \langle m_\alpha \cdot m_\beta, T_x f \rangle \\ &= \langle m_\beta \cdot m_\alpha, T_x f \rangle \\ &= \langle m_\beta, m_\alpha \cdot T_x f \rangle \\ &= \langle m_\beta, T_x(m_\alpha \cdot f) \rangle \\ &= \beta(x) \langle m_\beta, m_\alpha \cdot f \rangle \\ &= \beta(x) \langle m_\beta \cdot m_\alpha, f \rangle \quad (f \in X, x \in K). \end{aligned}$$

4 α -Amenability of Quotient Hypergroups

A closed nonempty subset H of K is called a subhypergroup if $H \cdot H = H$ and $\tilde{H} = H$, where $\tilde{H} := \{\tilde{x} : x \in H\}$. Let H be a subhypergroup of K . Then $K/H := \{xH : x \in K\}$ is a locally compact space with respect to the quotient topology. If H is a subgroup or a compact subhypergroup of K , then

$$\omega(xH, yH) := \int_K \delta_{zH} d\omega(x, y)(z) \quad (x, y \in K)$$

defines a hypergroup structure on K/H , which agrees with the double coset hypergroup $K//H$; see [12]. The properties and duals of subhypergroups and quotient of commutative hypergroups have been intensively studied by M. Voit [23, 24].

Theorem 4.1. Let H be a subgroup or a compact subhypergroup of K . Suppose $p : K \rightarrow K/H$ is the canonical projection, and $\widehat{p} : \widehat{K/H} \rightarrow \widehat{K}$ is defined by $\gamma \mapsto \gamma op$. Then K/H is γ -amenable if and only if K is γop -amenable.

Proof: Let K/H be γ -amenable. Then there exists a $M_\gamma : C^b(K/H) \rightarrow \mathbb{C}$ such that $M_\gamma(\gamma) = 1$, and $M_\gamma(T_{xH}f) = \gamma(xH)M_\gamma(f)$.

Since H is amenable [20], let m_1 be a mean on $C^b(H)$. For $f \in C^b(K)$, define

$$f^1 : K \rightarrow \mathbb{C} \quad \text{by} \quad f^1(x) := \langle m_1, T_x f|_H \rangle.$$

The function f^1 is continuous, bounded, and since m_1 is a mean for H , we have

$$\begin{aligned} T_h f^1(x) &= \int_K f^1(t) d\omega(h, x)(t) \\ &= \int_K \langle m_1, T_t f|_H \rangle d\omega(h, x)(t) \\ &= \langle m_1, \int_K T_t f|_H d\omega(h, x)(t) \rangle \\ &= \langle m_1, T_h [T_x f|_H] \rangle = \langle m_1, T_x f|_H \rangle = f^1(x), \end{aligned}$$

for all $h \in H$. Then according to the assumptions on H , [24, Lemma 1.5] implies that f^1 is constant on the cosets of H in K . We may write $f^1 = F \circ f$, $F \in C^b(K/H)$. Define

$$m : C^b(K) \rightarrow \mathbb{C} \quad \text{by} \quad m(f) = \langle M_\gamma, F \rangle.$$

We have

$$x_H F(yH) = T_x f^1(y) = \int_K \langle m_1, T_u f|_H \rangle d\omega(x, y)(u) = \langle m_1, T_y (T_x f)|_H \rangle,$$

as $u \mapsto T_u f|_H$ is continuous from K to $(C^b(H), \|\cdot\|_\infty)$ and the point evaluation functionals in $C^b(H)^*$ separates the points of $C^b(H)$; hence $T_{xH} F \circ p = (T_x f)^1$. Therefore,

$$m(T_x f) = \langle M_\gamma, T_{xH} F \rangle = \gamma(xH) \langle M_\gamma, F \rangle = \alpha(x) m(f).$$

Moreover, $\langle m, \alpha \rangle = \langle M_\gamma, \gamma \rangle = 1$, where $\gamma op = \alpha$. Then $m(T_x f) = \alpha(x)m(f)$ for all $f \in C^b(K)$ and $x \in K$.

To show the converse, let m_α be a α -mean on $C^b(K)$, and define

$$M : C^b(K/H) \longrightarrow \mathbb{C} \quad \text{by} \quad \langle M, f \rangle = \langle m_\alpha, f op \rangle.$$

Since

$$T_{xH} f(yH) = T_{xH} f op(y) = \int_{K/H} f d\omega(xH, yH) = \int_K f op d\omega(x, y) = T_x f op(y),$$

so $M(T_{xH} f) = \langle m_\alpha, T_x f op \rangle = \alpha(x) \langle m_\alpha, f op \rangle = \alpha(x) \langle M, f \rangle$. Since \widehat{p} is an isomorphism, [23, Theorem 2.5], and $\gamma op = \alpha$, we have

$$\langle M, \gamma \rangle = \langle m_\alpha, \gamma op \rangle = \langle m_\alpha, \alpha \rangle = 1.$$

Therefore, M is the desired γ -mean on $C^b(K/H)$. \square

4.2. Example: Let H be compact and H' be discrete commutative hypergroups. Let $K := H \vee H'$ denotes the joint hypergroup that H is a subhypergroup of K and $K/H \cong H'$ [25]. The hypergroups K , H and H' are amenable, and H is β -amenable for every $\beta \in \widehat{H}$ [7]. By Theorem 4.1, K is α -amenable if and only if H' is γ -amenable ($\alpha = \gamma op$).

Remark 4.3. Let G be a $[FIA]_B$ -group [16]. Then the space G_B , B -orbits in G , forms a hypergroup [12, 8.3]. If G is an amenable $[FIA]_B$ -group, then the hypergroup G_B is amenable [20, Corollary 3.11]. However, G_B may not be α -amenable for $\alpha \in \widehat{G}_B \setminus \{1\}$. For example, let $G := \mathbb{R}^n$ and B be the group of rotations which acts on G . Then the hypergroup $K := G_B$ can be identified with the Bessel-Kingman hypergroup $\mathbb{R}_0 := [0, \infty)$ of order $\nu = \frac{n-2}{2}$. Theorem 5.5 will show that if $n \geq 2$ then \mathbb{R}_0 is α -amenable if and only if $\alpha = 1$. Observe that $L^1(\mathbb{R}_0)$ is an amenable Banach algebra for $n = 1$ [26]; hence every maximal ideal of $L^1(\mathbb{R}_0)$ has a b.a.i. [4], consequently G_B is α -amenable for every $\alpha \in \widehat{G}_B$ [7].

Let K and H be hypergroups with left Haar measures. Then it is straightforward to show that $K \times H$ is a hypergroup with a left Haar measure. If K and H are commutative hypergroups, then $K \times H$ is a commutative hypergroup with a Haar measure. As in the case of locally compact groups [5], we have the following isomorphism

$$\phi : L^\infty(K) \times L^\infty(H) \longrightarrow L^\infty(K \times H) \quad \text{by} \quad (f, g) \longrightarrow \phi_{(f, g)}, \quad (3)$$

where $\phi_{(f, g)}(x, y) = f(x)g(y)$ for all $(x, y) \in K \times H$. Let $(x', y') \in K \times H$ and $(f, g) \in L^\infty(K) \times L^\infty(H)$. Then

$$\begin{aligned} T_{(x', y')} \phi_{(f, g)}(x, y) &= \int_{K \times H} \phi_{(f, g)}(t, t') d\omega(x', x) \times \omega(y', y)(t, t') \\ &= \int_K \int_H f(t)g(t') d\omega(x', x)(t) d\omega(y', y)(t') \\ &= T_{x'} f(x) T_{y'} g(y) \\ &= \phi_{(T_{x'} f, T_{y'} g)}(x, y). \end{aligned}$$

Theorem 4.4. Let K and H be commutative hypergroups. Then

- (i) the map ϕ defined in (3) is a homeomorphism between $\widehat{K} \times \widehat{H}$ and $\widehat{K \times H}$, where the dual spaces bear the compact-open topologies.
- (ii) $K \times H$ is $\phi_{(\alpha, \beta)}$ -amenable if and only if K and H are α and β -amenable respectively.

Proof: (i) It is the special case of [4, Proposition 19].

(ii) As in the case of locally compact groups [5], we have $L^1(K \times H) \cong L^1(K) \otimes_p L^1(H)$, where \otimes_p denotes the projection tensor product of two Banach algebras. If K is α -amenable and H is β -amenable, then $I(\alpha)$ and $I(\beta)$, the maximal ideals of $L^1(K)$ and $L^1(H)$ respectively, have b.a.i. [7]. Since $L^1(K)$ and $L^1(H)$ have b.a.i., $L^1(K) \otimes_p I(\beta) + I(\alpha) \otimes_p L^1(H)$, the maximal ideal in $L^1(K \times H)$ associated to the character $\phi_{(\alpha, \beta)}$, has a b.a.i. [5, Proposition 2.9.21], that equivalently $K \times H$ is $\phi_{(\alpha, \beta)}$ -amenable.

To prove the converse, suppose $m_{(\alpha, \beta)}$ is a $\phi_{(\alpha, \beta)}$ -mean on $L^\infty(K \times H)$, define

$$m_\alpha : L^\infty(K) \longrightarrow \mathbb{C} \text{ by } \langle m_\alpha, f \rangle := \langle m_{(\alpha, \beta)}, \phi_{(f, \beta)} \rangle.$$

We have

$$\begin{aligned} \langle m_\alpha, T_x f \rangle &= \langle m_{(\alpha, \beta)}, \phi_{(T_x f, \beta)} \rangle \\ &= \langle m_{(\alpha, \beta)}, T_{(x, e)} \phi_{(f, \beta)} \rangle \\ &= \alpha(x) \beta(e) \langle m_{(\alpha, \beta)}, \phi_{(f, \beta)} \rangle \\ &= \alpha(x) \langle m_{(\alpha, \beta)}, \phi_{(f, \beta)} \rangle \\ &= \alpha(x) \langle m_\alpha, f \rangle, \end{aligned}$$

for all $f \in L^\infty(K)$ and $x \in K$. Since $\langle m_\alpha, \alpha \rangle = \langle m_{(\alpha, \beta)}, \phi_{(\alpha, \beta)} \rangle = 1$, K is α -amenable. Similarly $m_\beta : L^\infty(H) \longrightarrow \mathbb{C}$ defined by $m_\beta(g) := \langle m_{(\alpha, \beta)}, \phi_{(\alpha, g)} \rangle$ is a β -mean on $L^\infty(H)$, hence H is β -amenable. \square

Remark 4.5. The proof of the previous theorem may also follow from Theorem 3.2 with a modifying [20, Proposition 3.8].

5 α -Amenability of Sturm-Liouville Hypergroups

Suppose $A : \mathbb{R}_0 \rightarrow \mathbb{R}$ is continuous, positive, and continuously differentiable on $\mathbb{R}_0 \setminus \{0\}$. Moreover, assume that

$$\frac{A'(x)}{A(x)} = \frac{\gamma_0(x)}{x} + \gamma_1(x), \quad (4)$$

for all x in a neighbourhood of 0, with $\gamma_0 \geq 0$ such that

SL1 one of the following additional conditions holds.

SL1.1 $\gamma_0 > 0$ and $\gamma_1 \in C^\infty(\mathbb{R})$, γ_1 being an odd function, or

SL1.2 $\gamma_0 = 0$ and $\gamma_1 \in C^1(\mathbb{R}_0)$.

SL2 There exists $\eta \in C^1(\mathbb{R}_0)$ such that $\eta(0) \geq 0$, $\frac{A'}{A} - \beta$ is nonnegative and decreasing on $\mathbb{R}_0 \setminus \{0\}$, and $q := \frac{1}{2}\eta' - \frac{1}{4}\eta^2 + \frac{A'}{2A}\eta$ is decreasing on $\mathbb{R}_0 \setminus \{0\}$.

The function A is called Chébli-Trimèche if A is a Sturm-Liouville function of type SL1.1 satisfying the additional assumptions that the quotient $\frac{A'}{A} \geq 0$ is decreasing and that A is increasing with $\lim_{x \rightarrow \infty} A(x) = \infty$. In this case SL2 is fulfilled with $\eta := 0$.

Let A be a Sturm-Liouville function satisfying (4) and SL2. Then there exists always a unique commutative hypergroup structure on \mathbb{R}_0 such that $A(x)dx$ is the Haar measure. A hypergroup established by this way is called a Sturm-Liouville hypergroup and it will be denoted by $(\mathbb{R}_0, A(x)dx)$. If A is a Chébli-Trimèche function, then the hypergroup $(\mathbb{R}_0, A(x)dx)$ is called Chébli-Trimèche hypergroup. The characters of $(\mathbb{R}_0, A(x)dx)$ can be considered as solution φ_λ of the differential equation

$$\left(\frac{d^2}{dx^2} + \frac{A'(x)}{A(x)} \frac{d}{dx} \right) \varphi_\lambda = -(\lambda^2 + \rho^2) \varphi_\lambda, \quad \varphi_\lambda(0) = 1, \quad \varphi'_\lambda(0) = 0,$$

where $\rho := \lim_{x \rightarrow \infty} \frac{A'(x)}{2A(x)}$, and $\lambda \in \mathbb{R}_\rho := \mathbb{R}_0 \cup i[0, \rho]$; see [3, Proposition 3.5.49]. As shown in [3, Sec.3.5], φ_0 is a strictly positive character, and φ_λ has the following integral representation

$$\varphi_\lambda(x) = \varphi_0(x) \int_{-x}^x e^{-i\lambda t} d\mu_x(t) \quad (5)$$

where $\mu_x \in M^1([-x, x])$ for every $x \in \mathbb{R}_0$ and all $\lambda \in \mathbb{C}$. In the particular case $\lambda := i\rho$, the equality (5) yields $|\varphi_0(x)| \leq e^{-\rho x}$, as $\varphi_{i\rho} = 1$.

Proposition 5.1. Let φ_λ be as above. Then

$$\left| \frac{d^n}{d\lambda^n} \varphi_\lambda(x) \right| \leq x^n e^{(|Im\lambda| - \rho)x}$$

for all $x \geq 0$, $\lambda \in \mathbb{C}$ and $n \in \mathbb{N}$.

Proof: For all $\lambda \in \mathbb{C} \setminus \{0\}$ and $x > 0$, applying the Lebesgue dominated convergence theorem [11] yields

$$\frac{d^n}{d\lambda^n} \varphi_\lambda(x) = \varphi_0(x) \int_{-x}^x (-it)^n e^{-i\lambda t} d\mu_x(t),$$

hence

$$\left| \frac{d^n}{d\lambda^n} \varphi_\lambda(x) \right| \leq \varphi_0(x) x^n \int_{-x}^x |e^{-i\lambda t}| d\mu_x(t)$$

which implies that $\left| \frac{d^n}{d\lambda^n} \varphi_\lambda(x) \right| \leq x^n e^{(|Im\lambda| - \rho)x}$. \square

To study the α -amenability of Sturm-Liouville hypergroups, we may require the following fact in general. The functional $D \in L^1(K)^*$ is called a α -derivation ($\alpha \in \widehat{K}$) on $L^1(K)$ if

$$D(f * g) = \widehat{f}(\alpha) D_\alpha(g) + \widehat{g}(\alpha) D_\alpha(f) \quad (f, g \in L^1(K)).$$

Observe that if the maximal ideal $I(\alpha)$ has an approximate identity, then $D|_{I(\alpha)} = 0$.

Lemma 5.2. Let $\alpha \in \widehat{K}$. If K is α -amenable, then every α -derivation on $L^1(K)$ is zero.

Proof: Let $D \in L^1(K)^*$ be a α -derivation on $L^1(K)$. Since $I(\alpha)$ has a b.a.i.[7], $D|_{I(\alpha)} = 0$. Assume that $g \in L^1(K)$ with $\widehat{g}(\alpha) = 1$. Consequently, $g * g - g \in I(\alpha)$ which implies that $D(g) = 0$. \square

Theorem 5.3. Let K be the Sturm-Liouville hypergroup with $\rho > 0$ and $\varphi_\lambda \in \widehat{K}$. Then K is φ_λ -amenable if and only if $\lambda = i\rho$.

Proof: By Proposition 5.1, the mapping

$$D_{\lambda_0} : L^1(K) \longrightarrow \mathbb{C}, \quad D_{\lambda_0}(f) = \frac{d}{d\lambda} \widehat{f}(\lambda) \Big|_{\lambda=\lambda_0} (\lambda_0 \neq i\rho),$$

is a well-defined bounded nonzero φ_{λ_0} -derivation. In that \mathbb{R}_0 is amenable [20], applying Lemma 5.2 will indicate that K is φ_λ -amenable if and only if $\lambda = i\rho$. \square

Remark 5.4. (i) Let K be the Sturm-Liouville hypergroup with $\rho > 0$. By the previous theorem, $L^1(K)$ is not weakly amenable as well as $\{\varphi_\lambda\}$, $\lambda \neq i\rho$, is not a spectral set.

Theorem 5.5 will show that the maximal ideals associated to the spectral sets do not have b.a.i. necessarily.

(ii) A Sturm-Liouville hypergroup is of exponential growth if and only if $\rho > 0$ [3, Proposition 3.5.55]. Then by Theorem 5.3 such hypergroups are φ_λ -amenable if and only if $\lambda = i\rho$. However, for $\rho = 0$ we do not have a certain assertion.

We now study special cases of Sturm-Liouville hypergroups in more details.

(i) **Bessel-Kingman hypergroup**¹

The Bessel-Kingman hypergroup is a Chébli-Trimèche hypergroup on \mathbb{R}_0 with $A(x) = x^{2\nu+1}$ when $\nu \geq -\frac{1}{2}$. The characters are given by

$$\alpha_\lambda^\nu(x) := 2^\nu \Gamma(\nu + 1) J_\nu(\lambda x) (\lambda x)^{-\nu},$$

where $J_\nu(x)$ is the Bessel function of order ν , and $\lambda \in \mathbb{R}_0$ represents the characters. The dual space \mathbb{R}_0 has also a hypergroup structure and the bidual space coincides with the hypergroup \mathbb{R}_0 [3]. As shown in [26], the L^1 -algebra of (\mathbb{R}_0, dx) , the Bessel-Kingman hypergroup of order $-\frac{1}{2}$, is amenable; as a result, (\mathbb{R}_0, dx) is α_λ^ν -amenable for every $\lambda \in \mathbb{R}_0$.

Suppose $L^1_{rad}(\mathbb{R}^n)$ is the subspace of $L^1(\mathbb{R}^n)$ of radial functions and $\nu = \frac{n-2}{2}$. It is a closed self-adjoint subalgebra of $L^1(\mathbb{R}^n)$ which is isometrically $*$ -isomorphic to the hypergroup algebra $L^1(\mathbb{R}_0, dm_n)$, where $dm_n(r) = \frac{2\pi^{d/2}}{\Gamma(n/2)} r^{n-1} dr$.

Theorem 5.5. Let K be the Bessel-Kingman hypergroup of order $\nu \geq 0$. If $\nu = 0$ or $\nu \geq \frac{1}{2}$, then K is α_λ^ν -amenable if and only if $\alpha_\lambda^\nu = 1$.

¹This subsection is from parts of the author's Ph.D. thesis at the Technical University of Munich.

Proof: (i) Let $\nu = 0$ and $\alpha_\lambda^0 \in \widehat{K}$. Since K is commutative, K is (1-)amenable [20]. Suppose now $\alpha_\lambda^0 \neq 1$ and K is α_λ^0 -amenable, so $I(\alpha_\lambda^0)$ has a b.a.i. If $I_r(\alpha_\lambda^0)$ is the corresponding ideal to $I(\alpha_\lambda^0)$ in $L_{rad}^1(\mathbb{R}^2)$, then $I_r(\alpha_\lambda^0)$ has a b.a.i., say $\{e'_i\}$. Let $I := [I_r(\alpha_\lambda^0) * L^1(\mathbb{R}^2)]^{cl}$. The group \mathbb{R}^2 is amenable [17], so let $\{e_i\}$ be a b.a.i for $L^1(\mathbb{R}^2)$. For every $f \in I_r(\alpha_\lambda^0)$ and $g \in L^1(\mathbb{R}^2)$, we have

$$\|f * g - (f * g) * (e'_i * e_i)\|_1 \leq \|g\|_1 \|f * e'_i - f\|_1 + \|f * e'_i\|_1 \|g - g * e_i\|_1.$$

The latter shows that $\{e'_i * e_i\}$ is a b.a.i for the closed ideal I . In [18, Theorem 17.2], it is shown that $Co(I)$, cospectrum of I , is a finite union of lines and points in \mathbb{R}^2 . But this contradicts the fact that $Co(I)$ is a circle with radius λ in \mathbb{R}^2 ; hence, K is α_λ^0 -amenable if and only if $\alpha_\lambda^0 = 1$.

(ii) Following [19], $\widehat{f}(\lambda) = \int_0^\infty f(x)\alpha_\lambda^\nu(x)dm(x)$ is differentiable, for all $f \in L^1(K)$, $\nu \geq \frac{1}{2}$, and $\lambda \neq 0$. Since $\frac{d}{dx}(x^{-\nu}J_\nu(x)) = -x^\nu J_{\nu+1}(x)$ and $J_\nu(x) = \mathcal{O}(x^{-1/2})$ as $x \rightarrow \infty$ [1], there exists a constant $A_\nu(\lambda_0) > 0$ such that $\left| \frac{d}{d\lambda} \widehat{f}(\lambda) \right|_{\lambda=\lambda_0} \leq A_\nu(\lambda_0) \|f\|_1$ ($\lambda_0 \neq 0$). Hence, the mapping

$$D_{\lambda_0} : L^1(\mathbb{R}_0^\nu, x^{2\nu+1}dx) \longrightarrow \mathbb{C}, \quad D_{\lambda_0}(f) = \frac{d}{d\lambda} \widehat{f}(\lambda) \Big|_{\lambda=\lambda_0},$$

is a well-defined bounded nonzero α_λ^ν -derivation. Hence, Lemma 5.2 implies that K is α_λ^ν -amenable if and only if $\alpha_\lambda^\nu = 1$. \square

(ii) Jacobi hypergroup of noncompact type

The Jacobi hypergroup of noncompact type is a Chébli-Trimèche hypergroup with

$$A^{(\alpha, \beta)}(x) := 2^{2\rho} \sinh^{2\alpha+1}(x) \cdot \cosh^{2\beta+1}(x),$$

where $\rho = \alpha + \beta + 1$ and $\alpha \geq \beta \geq -\frac{1}{2}$. The characters are given by Jacobi functions of order (α, β) , $\varphi_\lambda^{(\alpha, \beta)}(t) := {}_2F_1(\frac{1}{2}(\rho + i\lambda), \frac{1}{2}(\rho - i\lambda); \alpha + 1; -\text{sh}^2 t)$, where ${}_2F_1$ denotes the Gaussian hypergeometric function, $\alpha \geq \beta \geq -\frac{1}{2}$, $t \in \mathbb{R}_0$ and λ is the parameter of character $\varphi_\lambda^{(\alpha, \beta)}$ which varies on $\mathbb{R}_0 \cup [0, \rho]$. It is straightforward to show that $\rho = \alpha + \beta + 1$. As we have seen, if $\rho > 0$, then \mathbb{R}_0 is $\varphi_\lambda^{(\alpha, \beta)}$ -amenable if and only if $\varphi_\lambda^{(\alpha, \beta)} = 1$. If $\rho = 0$, then $\alpha = \beta = -\frac{1}{2}$, hence $\varphi_\lambda^{(-\frac{1}{2}, -\frac{1}{2})}(t) = \cos(\lambda t)$ so that turns \mathbb{R}_0 to the Bessel-Kingman hypergroup of order $\nu = -\frac{1}{2}$, which is $\varphi_\lambda^{(\alpha, \beta)}$ -amenable for every λ .

Let $A(\alpha, \beta) := \left\{ \check{\varphi} : \varphi \in L^1(\widehat{\mathbb{R}_0}, \pi) \right\}$. By the inverse theorem, [3, Theorem 2.2.32], we have $A(\alpha, \beta) \subseteq C_0([0, \infty))$. There exists a convolution structure on \mathbb{R}_0 such that π is the Haar measure on \mathbb{R}_0 and $A(\alpha, \beta)$ is a Banach algebra of functions on $[0, \infty)$; see [9]. The following estimation with Lemma 5.2 will indicate that for $\alpha \geq \frac{1}{2}$ and $\alpha \geq \beta \geq -\frac{1}{2}$, the maximal ideals of $A(\alpha, \beta)$ related to the points in $(0, \infty)$ do not have b.a.i.

Theorem 5.6. [15] For $\alpha \geq \frac{1}{2}$, $\alpha \geq \beta \geq -\frac{1}{2}$ and $\varepsilon > 0$ there is a constant $k > 0$ such that if $f \in A(\alpha, \beta)$ then $f|_{[\varepsilon, \infty)} \in C^{[\alpha+\frac{1}{2}]}([\varepsilon, \infty))$ and

$$\sup_{t \geq \varepsilon} |f^{(j)}(t)| \leq k \|f\|_{(\alpha, \beta)}, \quad 0 \leq j \leq [\alpha + \frac{1}{2}].$$

Remark 5.7. Here are the special cases of Jacobi hypergroups of noncompact type which are 1-amenable only:

- (i) Hyperbolic hypergroups, if $\beta = -\frac{1}{2}$ and $\rho = \alpha + \frac{1}{2} > 0$.
- (ii) Naimark hypergroup, if $\beta = -\frac{1}{2}$ and $\alpha = \frac{1}{2}$.

(iii) **Square hypergroup**

The Square hypergroup is a Sturm-Liouville hypergroup on \mathbb{R}_0 with $A(x) = (1+x)^2$ for all $x \in \mathbb{R}_0$. Obviously we have $\rho = 0$ and the characters are given by

$$\varphi_\lambda(x) := \begin{cases} \frac{1}{1+x} (\cos(\lambda x) + \frac{1}{\lambda} \sin(\lambda x)) & \text{if } \lambda \neq 0 \\ 1 & \text{if } \lambda = 0. \end{cases}$$

If $\lambda \neq 0$, then

$$\frac{d}{d\lambda} \varphi_\lambda(x) = \frac{x}{1+x} \left(-\sin(\lambda x) + \frac{\cos(\lambda x)}{\lambda} - \frac{1}{x\lambda^2} \sin(\lambda x) \right),$$

which is bounded as x varies. Therefore, the mapping

$$D_{\lambda_0} : L^1(\mathbb{R}_0, Adx) \longrightarrow \mathbb{C}, \quad D_{\lambda_0}(f) := \frac{d}{d\lambda} \widehat{f}(\lambda) \Big|_{\lambda=\lambda_0},$$

is a well-defined bounded nonzero φ_λ -derivation on $L^1(\mathbb{R}_0, Adx)$. Applying Lemma 5.2 results that \mathbb{R}_0 is 1-amenable only.

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